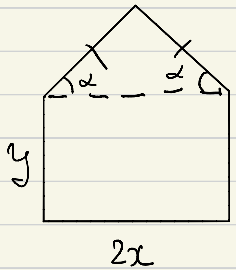
① Assignment 4 Q5



Let $2P$ be the perimeter

Then, $2P = 2x + 2y + \frac{2x}{\cos\alpha}$

$$P = x + y + \frac{x}{\cos\alpha} \quad \text{——} ①$$

$$\text{Area } A = 2xy + x^2\tan\alpha \quad \text{——} ②$$

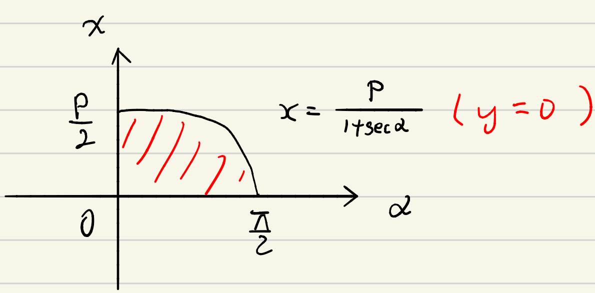
By ①, $\quad y = P - x - \frac{x}{\cos\alpha}$

$$A(x,\alpha) = 2x\left(P - x - \frac{x}{\cos\alpha}\right) + x^2\tan\alpha$$

We want to maximize $A(x,\alpha)$ subject to some constraints of x and α.

① $\quad 0 \le x \le \frac{P}{2}$, $\quad 0 \le \alpha \le \frac{\pi}{2}$

② $\quad 0 \le P - x - \frac{x}{\cos\alpha} \le P$



Boundary case :

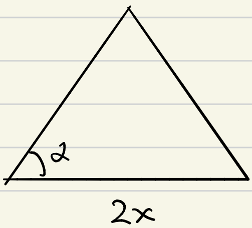
① $x=0$: area $= 0$

② $\alpha = 0$: Area $= 2x(p-2x)$

Maximized when $x = \frac{p}{4}$

$$A\left(\frac{p}{4}, 0\right) = \frac{p^2}{4}$$

③ $x = \frac{p}{1+\sec\alpha}$ or $(y=0)$



Area is maximized when

$\alpha = \frac{\pi}{3}$, $x = \frac{p}{3}$

$$\text{Area} = \frac{1}{2}\left(\frac{2p}{3}\right)^2 \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{9}p^2$$

Compare with the area obtained at the critical point.

② Problem set 10 Q2 (Math 2010 D 19/20)

Find the maximum and minimum values of the function $f(x,y,z) = 4x - 7y + 6z$ subjected to the constraint $g(x,y,z) = x^2 + 7y^2 + 12z^2 = 104$.

Sol$^{\underline{n}}$:

Let $g_1(x,y,z) = x^2 + 7y^2 + 12z^2 - 104$

Using Lagrange's multiplier, solve

$$\begin{cases} \nabla f = \lambda \nabla g_1 \\ g_1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (4, -7, 6) = \lambda(2x, 14y, 24z) \\ g_1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{2}{\lambda}, \quad y = -\frac{1}{2\lambda}, \quad z = \frac{1}{4\lambda} \\ x^2 + 7y^2 + 12z^2 = 104 \end{cases}$$

$$\left(\frac{2}{\lambda}\right)^2 + 7\left(-\frac{1}{2\lambda}\right)^2 + 12\left(\frac{1}{4\lambda}\right)^2 = 104$$

$$\left(\frac{1}{\lambda}\right)^2 = 16$$

$$\lambda = \pm\frac{1}{4}$$

Since $g_1(x,y,z)=0$ is a closed and bounded set (an ellipsoid), f attains maximum and minimum there.

For $\lambda = \frac{1}{4}$, $(x,y,z) = (8,-2,1)$

$$f(8,-2,1) = 52$$

For $\lambda = -\frac{1}{4}$, $(x,y,z) = (-8,2,-1)$

$$f(-8,2,-1) = -52$$

So, f attains max. at $(8,-2,1)$ and attains min at $(-8,2,-1)$

③ Thomas' Calculus p.837 Q28.

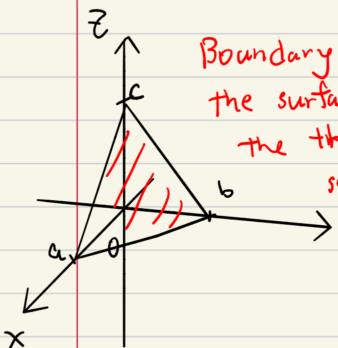
Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where $a>0$, $b>0$, and $c>0$.

Sol^n :

Maximize $V(x,y,z) = xyz$

subject to the constraints

$$\begin{cases} g(x,y,z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \\ x, y, z \geq 0 \end{cases}$$



Boundary of the surface is the three line segment

Apply Lagrange's multiplier,

$$\begin{cases} (yz, xz, xy) = \lambda\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \\ g(x,y,z) = 0 \end{cases}$$

For $\lambda \neq 0$, $x : y = xz : yz$
$= \frac{1}{b} : \frac{1}{a}$
$= a : b$

$y : z = xy : xz$
$= \frac{1}{c} : \frac{1}{b}$
$= b : c$

i.e. $x : y : z = a : b : c$

For $g(x,y,z) = 0$, it forces $(x,y,z) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

For $\lambda = 0$, $(x, y, z) = (a, 0, 0)$, $(0, b, 0)$ or $(0, 0, c)$

"This is the boundary case"

On the boundary of the surface,

$$V(x, y, z) = 0$$

$\therefore$ $V(x, y, z)$ attains maximum at $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.